Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE @ DIRECT®

An International Journal
computers & mathematics with applications

ELSEVIER Computers and Mathematics with Applications 46 (2003) 1633–1644
www.elsevier.com/locate/camwa

Properties of a Class of Multivalent Analytic Functions

J. PATEL
Department of Mathematics, Utkal University, Vani Vihar
Bhubaneswar - 751004, India
jpatelmath@sify.com

P. SAHOO
Department of Mathematics, Sonepur College
Sonepur - 767017, Orissa, India

(Received March 2001; accepted November 2002)

Abstract—We introduce a certain general class $\mathcal{V}_p^{\lambda}(a, c, A, B)$ of multivalent analytic functions in the open unit disc involving a linear operator. The object of the present is to investigate various properties and characteristics of this class by using the techniques of Briot-Bouquet differential subordination. © 2003 Elsevier Ltd. All rights reserved.

Keywords—Subordination, Hadamard product, Linear operator.

1. INTRODUCTION

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in \mathbb{N} = \{1, 2, 3, \dots\}), \tag{1.1}$$

which are analytic in the open unit disc $E = \{z : |z| < 1\}$. For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

For given arbitrary numbers A, B satisfying $-1 \le B < A \le 1$, we denote by $\mathcal{P}(A, B)$, the class of functions of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \tag{1.2}$$

that are analytic in E and satisfy the condition

$$p(z) \prec \frac{1 + Az}{1 + Bz} \qquad (z \in E).$$

0898-1221/03/$ - see front matter Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX
doi: 10.1016/S0898-1221(03)00390-0

(Here the symbol $\prec$ stands for subordination.) The class $\mathcal{P}(A,B)$ was investigated by Janowski [1].

For a function $f \in \mathcal{A}_p$ given by (1.1), the generalized Bernardi-Libera-Livingston integral operator $\mathcal{F}_\delta$ is defined by

$$\mathcal{F}_\delta(z) = \frac{\delta+p}{z^\delta}\int_0^z t^{\delta-1}f(t)\,dt = z^p + \sum_{n=1}^{\infty}\frac{\delta+p}{\delta+p+n}a_{p+n}z^{p+n} \qquad (\delta+p>0;\ z\in E). \tag{1.3}$$

It readily follows from (1.3) that

$$f\in\mathcal{A}_p \iff \mathcal{F}_\delta\in\mathcal{A}_p.$$

Furthermore, we have

$$\begin{aligned}\mathcal{H}_m(z) &= \mathcal{F}_{\delta_m}\left(\mathcal{F}_{\delta_{m-1}}\cdots(\mathcal{F}_{\delta_1}(z))\right)\\ &= z^p + \sum_{n=1}^{\infty}\left(\prod_{j=1}^{m}\frac{\delta_j+p}{\delta_j+p+n}\right)a_{p+n}z^{p+n} \qquad (p+\delta_j>0;\ j=1,2,3,\dots m).\end{aligned} \tag{1.4}$$

Let

$$\phi_p(a,c;z) = \sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}z^{p+n} \qquad (c\neq 0,-1,-2,\dots),$$

where $(x)_n$ denote the Pochhammer symbol defined by

$$(x)_n = \begin{cases} x(x+1)\cdots(x+n-1), & n\in\mathbb{N},\\ 1, & n=0.\end{cases}$$

We note that $\phi_p(a,1;z) = z^p/(1-z)^a$ and $\phi_1(2,1;z) = z/(1-z)^2$ is the Koebe function.

Corresponding to the function $\phi_p(a,c;z)$, Saitoh and Nunokawa [2] introduced a linear operator $\mathcal{L}_p(a,c)$ on $\mathcal{A}_p$ by

$$\mathcal{L}_p(a,c)f(z) = \phi_p(a,c;z) * f(z) \qquad (f\in\mathcal{A}_p).$$

If $f\in\mathcal{A}_p$ is given by (1.1), then

$$\mathcal{L}_p(a,c)f(z) = z^p + \sum_{n=1}^{\infty}\frac{(a)_n}{(c)_n}a_{p+n}z^{p+n} \qquad (c\neq 0,-1,-2,\dots;\ z\in E). \tag{1.5}$$

It follows from (1.5) that

$$z(\mathcal{L}_p(a,c)f(z))' = (c-1)\mathcal{L}_p(a,c-1)f(z) + (p+1-c)\mathcal{L}_p(a,c)f(z). \tag{1.6}$$

If $a\neq 0,-1,-2,\dots$, an application of the root test shows that the infinite series for $\mathcal{L}_p(a,c)f$ have the same radius of convergence as that of f because $\lim_{n\to\infty}|(a)_n/(c)_n|^{1/(n+p)} = 1$. Hence, $\mathcal{L}_p(a,c)$ maps $\mathcal{A}_p$ into itself. We note that for $f\in\mathcal{A}_p$,

(i) $\mathcal{L}_p(a,a)f(z) = f(z)$, $\mathcal{L}_p(p+1,p)f(z) = zf'(z)/p$.

(ii) $\mathcal{L}_p(\delta+p,1)f(z) = (z^p/(1-z)^{\delta+p}) * f(z) = \mathcal{D}^{\delta+p-1}f(z)$, where $\delta(>p)$ is any real number. In case of $p=1$ and $\delta=n\in\mathbb{N}$, $\mathcal{D}^\delta f$ is the Ruscheweyh derivative [3] of f.

(iii) $\mathcal{L}_p(\delta+p,\delta+p+1)f(z) = ((\delta+p)/z^\delta)\int_0^z t^{\delta-1}f(t)\,dt = \mathcal{F}_\delta(z)$, where $\delta+p>0$.

(iv) $\mathcal{L}_p(p+1,p+1-\mu)f(z) = \Gamma(p+1-\mu)z^\mu\mathcal{D}_z^\mu f(z)/\Gamma(p+1) = \mathcal{J}_z^{(\mu,p)}f(z)$ $(0\le\mu<1)$, where $\mathcal{D}_z^\mu f(z)$ is the fractional derivative of $f(z)$ of order μ, defined below in (1.9), with $\mathcal{D}_z^0 f(z) = f(z)$ and $\mathcal{D}_z^1 f(z) = f'(z)$.

Making use of the operator $\mathcal{L}_p(a,c)$, we now introduce a subclass of $\mathcal{A}_p$ as follows.

DEFINITION 1. *A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{V}_p^\lambda(a,c,A,B)$ ($\lambda \geq 0$, $a > 0$, $c > 1$, $-1 \leq B < A \leq 1$), iff it satisfies*

$$(1-\lambda)\frac{\mathcal{L}_p(a,c)f(z)}{z^p} + \lambda\frac{\mathcal{L}_p(a,c-1)f(z)}{z^p} \prec \frac{1+Az}{1+Bz} \qquad (z \in E). \tag{1.7}$$

By specializing the parameters λ, a, c, A, and B, we obtain the following subclasses of analytic functions studied by various authors:

(i) $\mathcal{V}_1^1(2,2,1-2\alpha,-1) = \mathcal{R}(\alpha)$ $(0 \leq \alpha < 1)$ *[4]*;

(ii) $\mathcal{V}_p^1(p+1,p+1,1,(1/\alpha)-1) = \mathcal{S}_p(\alpha)$ $(\alpha > 1/2)$ *[5]*;

(iii) $\mathcal{V}_1^1(2,2,\beta(1-2\alpha),-\beta) = \mathcal{V}(\alpha,\beta)$ $(0 \leq \alpha < 1,\ 0 < \beta \leq 1)$ *[6]*;

(iv) $\mathcal{V}_p^\lambda(p+1,p+1-\mu,1,\beta(1-2\alpha/p),-\beta) = \mathcal{V}_p^\lambda(\mu,\alpha,\beta)$;

where $\mathcal{V}_p^\lambda(\mu,\alpha,\beta)$ denotes the class of functions $f \in \mathcal{A}_p$ satisfying the condition

$$\left|\frac{(1-\lambda)\mathcal{J}_z^{(\mu,p)}f(z) + \lambda\mathcal{J}_z^{(1+\mu,p)}f(z) - z^p}{(1-\lambda)\mathcal{J}_z^{(\mu,p)}f(z) + \lambda\mathcal{J}_z^{(1+\mu,p)}f(z) + (1-(2\alpha/p))z^p}\right| < \beta \qquad (z \in E),$$

where $0 \leq \mu < 1$, $0 \leq \alpha < p$, and $0 < \beta \leq 1$.

Various operators of fractional calculus (i.e., fractional integral and fractional derivative) have been studied in the literature rather extensively. We find it to be convenient to restrict ourselves to the following definitions used recently by Owa [7].

DEFINITION 2. *The fractional integral operator of order μ is defined for a function f, by*

$$\mathcal{D}_z^{-\mu}f(z) = \frac{1}{\Gamma(\lambda)}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\mu}}\,d\zeta \qquad (\mu > 0), \tag{1.8}$$

where f is analytic in a simply-connected domain of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\mu-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

DEFINITION 3. *The fractional derivative of order μ is defined for a function f, by*

$$\mathcal{D}_z^{\mu}f(z) = \frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{\mu}}\,d\zeta \qquad (0 \leq \mu < 1), \tag{1.9}$$

where f is constrained, and the multiplicity of $(z-\zeta)^{-\mu}$ is removed, as in Definition 2.

DEFINITION 4. *Under the hypotheses of Definition 3, the fractional derivative of order $n+\mu$ is defined, for a function f, by*

$$\mathcal{D}_z^{n+\mu}f(z) = \frac{d^n}{dz^n}\left(\mathcal{D}_z^{\mu}f(z)\right) \qquad (0 \leq \mu < 1,\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{1.10}$$

2. PRELIMINARIES

In our present investigation of the general class $\mathcal{V}_p^\lambda(a,c,A,B)$, we shall require the following lemmas.

LEMMA 1. *Let $h(z)$ be a convex (univalent) in E with $h(0) = 1$, and let the function $\phi(z)$ given by (1.2) be analytic in E. If*

$$\phi(z) + \frac{z\phi'(z)}{\gamma} \prec h(z) \qquad (z \in E),$$

for $\gamma \neq 0$ and $\mathrm{Re}(\gamma) \geq 0$, then

$$\phi(z) \prec \psi(z) = \frac{\gamma}{z^\gamma}\int_0^z t^{\gamma-1}h(t)\,dt \prec h(z) \qquad (z \in E),$$

and $\psi(z)$ is the best dominant.

The above lemma is due to Miller and Mocanu [8].

We now state a result obtained by Singh and Singh [9].

LEMMA 2. *Let $p(z)$ be the be analytic in E, $p(0) = 1$, and $\operatorname{Re}\{p(z)\} > 1/2$ in E. Then for any function $F(z)$ analytic in E, the function $(p * F)(z)$ takes values in the convex hull of the image of E under $F(z)$.*

Let α_j $(j = 1, 2, \ldots, r)$ and β_j $(j = 1, 2, \ldots, s)$ be complex numbers with $\beta_j \neq 0, -1, -2, \ldots$ $(j = 1, 2, \ldots, s)$. Then the generalized hypergeometric function ${}_rF_s(z)$ is defined by

$$ {}_rF_s(z) \equiv {}_rF_s(\alpha_1, \alpha_2, \ldots, \alpha_r; \beta_1, \beta_2, \ldots, s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{z^n}{n!} \quad . \quad (r \leq s+1). \tag{2.1} $$

We note that the ${}_rF_s(z)$ series in (2.1) converges absolutely for $|z| < \infty$ if $r < s + 1$, and for $z \in E$ if $r = s + 1$.

The following identities are well known [10].

LEMMA 3. *For real or complex numbers α_1, α_2, and β_1 $(\beta_1 \neq 0, -1, -2, \ldots)$, we have*

$$ \begin{aligned} &\int_0^1 t^{\alpha_2 - 1}(1-t)^{\beta_1 - \alpha_2 - 1}(1 - tz)^{-\alpha_1}\, dt \\ &= \frac{\Gamma(\alpha_2)\Gamma(\beta_1 - \alpha_2)}{\Gamma(\beta_1)}\, {}_2F_1(\alpha_1, \alpha_2; \beta_1; z) \qquad (\operatorname{Re}(\beta_1) > \operatorname{Re}(\alpha_2) > 0), \end{aligned} \tag{2.2} $$

$$ {}_2F_1(\alpha_1, \alpha_2; \beta_1; z) = (1-z)^{-\alpha_1}\, {}_2F_1\left(\alpha_1, \beta_1 - \alpha_2; \beta_1; \frac{z}{z-1}\right), \tag{2.3} $$

$$ (\alpha_2 + 1)\, {}_2F_1(1, \alpha_2; \alpha_2 + 1; z) = (\alpha_2 + 1) + \alpha_2 z\, {}_2F_1(1, \alpha_2 + 1; \alpha_2 + 2; z), \tag{2.4} $$

$$ {}_2F_1\left(1, 1; 2; \frac{z}{z+1}\right) = \frac{z+1}{z} \ln(1+z) \qquad (z \neq 0). \tag{2.5} $$

3. MAIN RESULTS

THEOREM 1. *Let the function f defined by (1.1) be in the class $\mathcal{V}_p^\lambda(a, c, A, B)$, $\lambda > 0$. Then*

$$ \frac{\mathcal{L}_p(a,c)f(z)}{z^p} \prec \theta(z) \prec \frac{1 + Az}{1 + Bz} \qquad (z \in E), \tag{3.1} $$

where

$$ \theta(z) = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1 + Bz)^{-1}\, {}_2F_1\left(1, 1; \dfrac{c-1}{\lambda} + 1; \dfrac{Bz}{Bz+1}\right), & B \neq 0, \\ 1 + \dfrac{c-1}{c-1+\lambda} Az, & B = 0, \end{cases} $$

and is the best dominant of (3.1). Furthermore,

$$ \operatorname{Re}\left\{\frac{\mathcal{L}_p(a,c)f(z)}{z^p}\right\} > \kappa(\lambda, c, A, B) \qquad (z \in E), \tag{3.2} $$

where

$$ \kappa(\lambda, c, A, B) = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1 - B)^{-1}\, {}_2F_1\left(1, 1; \dfrac{c-1}{\lambda} + 1; \dfrac{B}{B-1}\right), & B \neq 0, \\ 1 - \dfrac{c-1}{c-1+\lambda} A, & B = 0. \end{cases} $$

The estimate in (3.2) is best possible.

PROOF. Consider the function $\phi(z)$ defined by

$$ \phi(z) = \frac{\mathcal{L}_p(a,c)f(z)}{z^p} \qquad (z \in E). \tag{3.3} $$

Then $\phi(z)$ is of the form (1.2) and is analytic in E. Differentiating both sides of (3.3) and using identity (1.6) in the resulting equation, we get

$$(1-\lambda)\frac{\mathcal{L}_p(a,c)f(z)}{z^p}+\lambda\frac{\mathcal{L}_p(a,c-1)f(z)}{z^p}\prec\phi(z)+\frac{z\phi'(z)}{(c-1)/\lambda}\prec\frac{1+Az}{1+Bz}\qquad(z\in E).$$

Now, by using Lemma 1 for $\gamma=(c-1)/\lambda$, we deduce that

$$\begin{aligned}\frac{\mathcal{L}_p(a,c)f(z)}{z^p}\prec\theta(z)&=\frac{c-1}{\lambda}z^{-(c-1)/\lambda}\int_0^z t^{(c-1)/\lambda-1}\left(\frac{1+At}{1+Bt}\right)dt\\&=\begin{cases}\dfrac{A}{B}+\left(1-\dfrac{A}{B}\right)(1+Bz)^{-1}\,{}_2F_1\left(1,1;\dfrac{c-1}{\lambda}+1;\dfrac{Bz}{Bz+1}\right), & B\neq 0,\\ 1+\dfrac{c-1}{c-1+\lambda}Az, & B=0,\end{cases}\end{aligned}$$

by change of variables followed by the use of identities (2.2)–(2.4). This proves assertion (3.1).

Next, in order to prove (3.2), it suffices to show that

$$\inf_{|z|<1}\{\mathrm{Re}(\theta(z))\}=\theta(-1).\tag{3.4}$$

For $|z|\le r<1$, we have

$$\mathrm{Re}\left\{\frac{1+Az}{1+Bz}\right\}\ge\frac{1-Ar}{1-Br}.$$

Setting

$$g(s,z)=\frac{(1+Asz)}{(1+Bsz)}\quad(0\le s\le 1)\qquad\text{and}\qquad d\mu(s)=\left(\frac{(c-1)}{\lambda}\right)s^{((c-1)/\lambda)-1}\,ds,$$

we get

$$\theta(z)=\int_0^1 g(s,z)\,d\mu(s),$$

so that

$$\mathrm{Re}\{\theta(z)\}\ge\int_0^1\left(\frac{1-Asr}{1-Bsr}\right)d\mu(s)=\theta(-r)\qquad(|z|\le r<1).$$

Letting $r\to 1^-$ in the above inequality, we obtain assertion (3.4).

The estimate in (3.2) is best possible as the function $\theta(z)$ is the best dominant of (3.1).

COROLLARY 1. *For $0\le\lambda_2<\lambda_1$, we have*

$$\mathcal{V}_p^{\lambda_1}(a,c,A,B)\subset\mathcal{V}_p^{\lambda_2}(a,c,A,B).$$

PROOF. Let $f\in\mathcal{V}_p^{\lambda_1}(a,c,A,B)$. Then by Theorem 1, we have $f\in\mathcal{V}_p^{0}(a,c,A,B)$. Since

$$\begin{aligned}&(1-\lambda_2)\frac{\mathcal{L}_p(a,c)f(z)}{z^p}+\lambda_2\frac{\mathcal{L}_p(a,c-1)f(z)}{z^p}\\&=\left(1-\frac{\lambda_2}{\lambda_1}\right)\frac{\mathcal{L}_p(a,c)f(z)}{z^p}+\frac{\lambda_2}{\lambda_1}\left\{(1-\lambda_1)\frac{\mathcal{L}_p(a,c)f(z)}{z^p}+\lambda_1\frac{\mathcal{L}_p(a,c-1)f(z)}{z^p}\right\}\\&\prec\frac{1+Az}{2+Bz}\qquad(z\in E),\end{aligned}$$

we see that $f\in\mathcal{V}_p^{\lambda_2}(a,c,A,B)$.

Taking $\lambda=1$, $a=p+1$, $c=p+1-\mu$, $A=1-2\alpha/p$, and $B=-1$ in Theorem 1, we get the following corollary.

COROLLARY 2. *Let the function f given by (1.1) satisfy*

$$\operatorname{Re}\left\{\frac{\mathcal{J}_z^{(\mu,p)}f(z)}{z^p}\right\} > \frac{\alpha}{p} \qquad (0 \le \alpha < p;\ z \in E).$$

Then

$$\operatorname{Re}\left\{\frac{\mathcal{J}_z^{(\mu,p)}f(z)}{z^p}\right\} > \frac{\alpha}{p} + \left(1-\frac{\alpha}{p}\right)\left\{{}_2F_1\left(1,1;p+1-\mu;\frac{1}{2}\right)-1\right\} \qquad (z \in E).$$

The result is best possible.

Putting $\mu = 0$ in Corollary 2, we have the following corollary.

COROLLARY 3. *If $f \in \mathcal{A}_p$ satisfies*

$$\operatorname{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha \qquad (0 \le \alpha < p;\ z \in E),$$

then

$$\operatorname{Re}\left\{\frac{f(z)}{z^p}\right\} > \frac{\alpha}{p} + \left(1-\frac{\alpha}{p}\right)\left\{{}_2F_1\left(1,1;p+1;\frac{1}{2}\right)-1\right\} \qquad (z \in E).$$

The result is best possible.

For $\lambda = 1$, $a = p+\delta$, $c = p+\delta+1$, $A = 1-2\alpha/p$, and $B = -1$ in Theorem 1, we obtain the following corollary.

COROLLARY 4. *If $f \in \mathcal{A}_p$ satisfies*

$$\operatorname{Re}\left\{\frac{f(z)}{z^p}\right\} > \frac{\alpha}{p} \qquad (0 \le \alpha < p;\ z \in E),$$

then the function $\mathcal{F}_\delta$ defined by (1.3) satisfies

$$\operatorname{Re}\left\{\frac{\mathcal{F}_\delta(z)}{z^p}\right\} > \frac{\alpha}{p} + \left(1-\frac{\alpha}{p}\right)\left\{{}_2F_1\left(1,1;p+\delta+1;\frac{1}{2}\right)-1\right\} \qquad (z \in E).$$

The result is best possible.

REMARK 1. We note that Corollary 4 improves the corresponding result obtained by Obradovič [11] for $p = 1$.

REMARK 2. If $f \in \mathcal{A}_p$ satisfies $\operatorname{Re}\{f'(z)/z^{p-1}\} > \alpha$ $(0 \le \alpha < p;\ z \in E)$, then with the aid of Corollaries 3 and 4, we deduce that

$$\begin{aligned}\operatorname{Re}\left\{\frac{\mathcal{F}_\delta(z)}{z^p}\right\} > \frac{\alpha}{p} + \left(1-\frac{\alpha}{p}\right)\Bigg[&\left({}_2F_1\left(1,1;p+1;\frac{1}{2}\right)-1\right)\\ &+\left({}_2F_1\left(1,1;p+\delta+1;\frac{1}{2}\right)-1\right)\left(2-{}_2F_1\left(1,1;p+1;\frac{1}{2}\right)\right)\Bigg],\end{aligned}$$

which improves a result due to Fukui *et al.* [4] for $p = 1$.

THEOREM 2. *If $f \in \mathcal{V}_p^0(a,c,A,B)$, then the function $\mathcal{F}_\delta$ defined by (1.3) satisfies*

$$\frac{\mathcal{L}_p(a,c)\mathcal{F}_\delta(z)}{z^p} \prec q(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in E), \tag{3.5}$$

where

$$q(z) = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1+Bz)^{-1}\, {}_2F_1\left(1, 1; p+\delta+1; \dfrac{Bz}{Bz+1}\right), & B \neq 0, \\ 1 + \dfrac{p+\delta}{p+\delta+1} Az, & B = 0, \end{cases}$$

and is the best dominant of (3.5). Furthermore,

$$\operatorname{Re}\left\{\frac{\mathcal{L}_p(a,c)\mathcal{F}_\delta(z)}{z^p}\right\} > \varkappa(\delta, p, A, B) \qquad (z \in E), \tag{3.6}$$

where

$$\varkappa(\delta, p, A, B) = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1-B)^{-1}\, {}_2F_1\left(1, 1; p+\delta+1; \dfrac{B}{B-1}\right), & B \neq 0, \\ 1 - \dfrac{p+\delta}{p+\delta+1} A, & B = 0. \end{cases}$$

The estimate in (3.6) is best possible.

PROOF. Let

$$\phi(z) = \frac{\mathcal{L}_p(a,c)\mathcal{F}_\delta(z)}{z^p} \qquad (z \in E). \tag{3.7}$$

Then $\phi(z)$ is analytic in E with $\phi(0) = 1$. Differentiating (3.7) and using the identity

$$z(\mathcal{L}_p(a,c)\mathcal{F}_\delta(z))' = (\delta+p)\mathcal{L}_p(a,c)f(z) - \delta\, \mathcal{L}_p(a,c)\mathcal{F}_\delta(z) \tag{3.8}$$

in the resulting equation, we obtain

$$\phi(z) + \frac{z\phi'(z)}{\delta+p} = \frac{\mathcal{L}_p(a,c)f(z)}{z^p} \prec \frac{1+Az}{1+Bz} \qquad (z \in E),$$

which, with the aid of Lemma 1 for $\gamma = \delta + p$, yields

$$\frac{\mathcal{L}_p(a,c)\mathcal{F}_\delta(z)}{z^p} \prec q(z) = (\delta+p)z^{-(\delta+p)} \int_0^z t^{\delta+p-1}\left(\frac{1+At}{1+Bt}\right) dt.$$

Assertion (3.5) and estimate (3.6) can now be deduced on the same lines as that of Theorem 1. This completes the proof of Theorem 2.

Taking $A = 1 - 2\alpha/p$ and $B = -1$ in Theorem 2, we get the following corollary.

COROLLARY 5. *If $f \in \mathcal{A}_p$ satisfies*

$$\operatorname{Re}\left\{\frac{\mathcal{L}_p(a,c)f(z)}{z^p}\right\} > \frac{\alpha}{p} \qquad (0 \le \alpha < p;\ z \in E), \tag{3.9}$$

then

$$\operatorname{Re}\left\{\frac{\mathcal{L}_p(a,c)\mathcal{F}_\delta(z)}{z^p}\right\} > \frac{\alpha}{p} + \left(1 - \frac{\alpha}{p}\right)\left\{{}_2F_1\left(1, 1; p+\delta+1; \frac{1}{2}\right) - 1\right\} \qquad (z \in E).$$

The result is best possible.

COROLLARY 6. *Under the hypothesis of Corollary 5, the function $\mathcal{H}_m(z)$ defined by (1.4) satisfies*

$$\operatorname{Re}\left\{\frac{\mathcal{L}_p(a,c)\mathcal{H}_m(z)}{z^p}\right\} > \frac{\rho_m}{p} \qquad (z \in E),$$

where $\rho_0 = \alpha$, and

$$\rho_j = \rho_{j-1} + (p - \rho_{j-1})\left\{{}_2F_1\left(1, 1; p+\delta_j+1; \frac{1}{2}\right) - 1\right\} \qquad (j = 1, 2, \dots, m).$$

The result is best possible.

Putting $a = p + 1$ and $c = p + 1 - \mu$ in Corollary 5, we have the following corollary.

COROLLARY 7. *If $f \in \mathcal{A}_p$ satisfies*

$$\operatorname{Re}\left\{\frac{\mathcal{J}_z^{(\mu,p)} f(z)}{z^p}\right\} > \frac{\alpha}{p} \qquad (0 \le \alpha < p,\ 0 \le \mu < p;\ z \in E),$$

then

$$\operatorname{Re}\left\{\frac{\mathcal{J}_z^{(\mu,p)} \mathcal{F}_\delta(z)}{z^p}\right\} > \frac{\alpha}{p} + \left(1 - \frac{\alpha}{p}\right)\left\{{}_2F_1\left(1,1;p+\delta+1;\frac{1}{2}\right) - 1\right\} \qquad (z \in E).$$

The result is best possible.

Taking $a = p+1$ and $c = p$ in Corollary 5, we get the following corollary which in turn improves the corresponding result of Fukui *et al.* [4] for $p = 1$.

COROLLARY 8. *If $f \in \mathcal{A}_p$ satisfies*

$$\operatorname{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha \qquad (0 \le \alpha < p;\ z \in E),$$

then

$$\operatorname{Re}\left\{\frac{\mathcal{F}_\delta'(z)}{z^{p-1}}\right\} > \alpha + (p - \alpha)\left\{{}_2F_1\left(1,1;p+\delta+1;\frac{1}{2}\right) - 1\right\} \qquad (z \in E).$$

The result is best possible.

THEOREM 3. *We have*

$$f \in \mathcal{V}_p^0(a,c,A,B) \Longleftrightarrow \mathcal{F}_{c-p-1} \in \mathcal{V}_p^1(a,c,A,B).$$

PROOF. Using identity (3.8) and

$$z(\mathcal{L}_p(a,c)\mathcal{F}_\delta(z))' = (c-1)\mathcal{L}_p(a,c-1)\mathcal{F}_\delta(z) + (p+1-c)\mathcal{L}_p(a,c)\mathcal{F}_\delta(z),$$

for $\delta = c - p - 1$, we deduce that

$$\mathcal{L}_p(a,c)f(z) = \mathcal{L}_p(a,c)\mathcal{F}_{c-p-1}(z)$$

and the assertion of Theorem 3 follows by using the definition of the class $\mathcal{V}_p^\lambda(a,c,A,B)$.

THEOREM 4. *If f, given by (1.1), belongs to the class $\mathcal{V}_p^\lambda(a,c,A,B)$, then*

$$|a_{p+n}| \le \frac{(A-B)(c-1)_{n+1}}{(c-1+\lambda n)(a)_n} \qquad (n \ge 1). \tag{3.10}$$

The estimate is sharp.

PROOF. Since $f \in \mathcal{V}_p^\lambda(a,c,A,B)$, we have

$$(1-\lambda)\frac{\mathcal{L}_p(a,c)f(z)}{z^p} + \lambda\frac{\mathcal{L}_p(a,c-1)f(z)}{z^p} = p(z), \tag{3.11}$$

where $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}(A,B)$. Substituting the power series expansion of $\mathcal{L}_p(a,c)f(z)$, $\mathcal{L}_p(a,c-1)f(z)$, and $p(z)$ in (3.11) and equating the coefficients of z^n on both the sides of the resulting equation, we obtain

$$\frac{(c-1+\lambda n)(a)_n}{(c-1)_{n+1}} a_{p+n} = p_n \qquad (n \ge 1). \tag{3.12}$$

Using the well-known [12] coefficient estimates

$$|p_n| \le A - B\; n \ge 1$$

in (3.12), we get the required estimate (3.10).

In order to establish the sharpness of (3.10), consider the functions $f_n(z)$ defined by

$$(1-\lambda)\,\frac{\mathcal{L}_p(a,c)f_n(z)}{z^p} + \lambda\,\frac{\mathcal{L}_p(a,c-1)f_n(z)}{z^p} = \frac{1+Az^n}{1+Bz^n} \qquad (n \ge 1).$$

Clearly, $f_n(z) \in \mathcal{V}_p^{\lambda}(a,c,A,B)$ for each $n \ge 1$. It is easy to see that the functions $f_n(z)$ have the expansion

$$f_n(z) = z^p + \frac{(A-B)(c-1)_{n+1}}{(c-1+\lambda n)(a)_n}\, z^{p+n} + \cdots$$

showing that the estimates in (3.10) are sharp.

THEOREM 5. *Let f, given by (1.1), belong to the class $\mathcal{V}_p^{\lambda}(a,c,A,B)$ and u be any complex number. Then*

$$\left|a_{p+2} - u\, a_{p+1}^2\right| \le \frac{(A-B)(c-1)_3}{(c-1+2\lambda)(a)_2} \max\left\{1, \left|B + u\,\frac{(A-B)(c-1+2\lambda)(a+1)(c-1)_2}{a(c+1)(c-1+\lambda)^2}\right|\right\}. \tag{3.13}$$

The estimate in (3.13) is sharp.

PROOF. From (1.7), we deduce that

$$\begin{aligned}&(1-\lambda)\,\frac{\mathcal{L}_p(a,c)f(z)}{z^p} + \lambda\,\frac{\mathcal{L}_p(a,c-1)f(z)}{z^p} - 1\\ &= \left[A - B\left\{(1-\lambda)\,\frac{\mathcal{L}_p(a,c)f(z)}{z^p} + \lambda\,\frac{\mathcal{L}_p(a,c-1)f(z)}{z^p}\right\}\right]\omega(z),\end{aligned} \tag{3.14}$$

where $\omega(z) = \sum_{n=1}^{\infty} w_n z^n$ is analytic in E and satisfies $|\omega(z)| \le |z|$ for $z \in E$. Substituting the power series expansion of $\mathcal{L}_p(a,c)f(z)$, $\mathcal{L}_p(a,c-1)f(z)$, and $\omega(z)$ in (3.14), and equating the coefficients of z and z^2, we get

$$a_{p+1} = \frac{(A-B)(c-1)_2}{a\,(c-1+\lambda)}\, w_1, \tag{3.15}$$

$$a_{p+2} = \frac{(A-B)(c-1)_3}{(a)_2\,(c-1+\lambda)}\left(w_2 - Bw_1^2\right). \tag{3.16}$$

It is well known [13] that for every complex number ν

$$\left|w_2 - \nu\, w_1^2\right| \le \max\{1, |\nu|\} \tag{3.17}$$

and the result is sharp for $\omega(z) = z$ and $\omega(z) = z^2$, respectively, for $|\nu| \ge 1$ and $|\nu| < 1$. From (3.15) and (3.16), we have

$$\left|a_{p+2} - u\, a_{p+1}^2\right| = \frac{(A-B)(c-1)_3}{(c-1+2\lambda)(a)_2}\left|w_2 - \nu\, w_1^2\right|, \tag{3.18}$$

where

$$\nu = B + u\,\frac{(A-B)(c-1+2\lambda)(a+1)(c-1)_2}{a(c+1)(c-1+\lambda)^2}.$$

Now, by using (3.17) in (3.18), we get the required result. Result (3.13) is sharp as stimate (3.17) is sharp.

THEOREM 6. *Let* $f \in \mathcal{V}_p^\lambda(a,c,A,B)$ *and* $g \in \mathcal{A}_p$ *with* $\mathrm{Re}\{g(z)/z^p\} > 1/2$ *for* $z \in E$. *Then* $h = f * g \in \mathcal{V}_p^\lambda(a,c,A,B)$.

PROOF. We can write

$$\begin{aligned}&(1-\lambda)\frac{\mathcal{L}_p(a,c)h(z)}{z^p} + \lambda\frac{\mathcal{L}_p(a,c-1)h(z)}{z^p}\\ &\quad= \left\{(1-\lambda)\frac{\mathcal{L}_p(a,c)f(z)}{z^p} + \lambda\frac{\mathcal{L}_p(a,c-1)f(z)}{z^p}\right\} * \frac{g(z)}{z^p}.\end{aligned} \tag{3.19}$$

Since $\mathrm{Re}\{g(z)/z^p\} > 1/2$ in E and $f \in \mathcal{V}_p^\lambda(a,c,A,B)$, it follows from (3.19) and Lemma 3 that $h \in \mathcal{V}_p^\lambda(a,c,A,B)$. This completes the proof of Theorem 6.

COROLLARY 9. *Let* $f \in \mathcal{V}_p^\lambda(a,c,A,B)$ *and* $g \in \mathcal{A}_p$ *satisfy*

$$\mathrm{Re}\left\{(1-\mu)\frac{g(z)}{z^p} + \mu\frac{g'(z)}{pz^{p-1}}\right\} > \frac{3 - 2\,{}_2F_1(1,1;p/\mu+1;1/2)}{2\left\{2 - {}_2F_1(1,1;p/\mu+1;1/2)\right\}} \qquad (0<\mu;\ z\in E). \tag{3.20}$$

Then $f * g \in \mathcal{V}_p^\lambda(a,c,A,B)$.

PROOF. From Theorem 1 (for $a = c = p+1$, $\lambda = \mu > 0$, $A = 0$, and $B = -1$), condition (3.20) implies

$$\mathrm{Re}\left\{\frac{g(z)}{z^p}\right\} > \frac{1}{2} \qquad (z \in E).$$

Using this, it follows from Theorem 6 that $f * g \in \mathcal{V}_p^\lambda(a,c,A,B)$.

THEOREM 7. *If each of the functions* $f(z)$ *given by (1.1) and* $g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n}z^{p+n}$ *belongs to the class* $\mathcal{V}_p^\lambda(a,c,A,B)$, *then so does the function* $h(z) = (1-\lambda)\mathcal{L}_p(a,c)(f*g)(z) + \lambda\mathcal{L}_p(a,c-1)(f*g)(z)$.

PROOF. Since $f \in \mathcal{V}_p^\lambda(a,c,A,B)$, it follows by (3.14) that

$$\begin{aligned}&\left|(1-\lambda)\frac{\mathcal{L}_p(a,c)f(z)}{z^p} + \lambda\frac{\mathcal{L}_p(a,c-1)f(z)}{z^p} - 1\right|\\ &\qquad< \left|A - B\left\{(1-\lambda)\frac{\mathcal{L}_p(a,c)f(z)}{z^p} + \lambda\frac{\mathcal{L}_p(a,c-1)f(z)}{z^p}\right\}\right|,\end{aligned}$$

which is equivalent to

$$\left|(1-\lambda)\frac{\mathcal{L}_p(a,c)f(z)}{z^p} + \lambda\frac{\mathcal{L}_p(a,c-1)f(z)}{z^p} - \xi\right| < \eta \qquad (z \in E), \tag{3.21}$$

where

$$\xi = \frac{1-AB}{1-B^2} \quad \text{and} \quad \eta = \frac{A-B}{1-B^2}.$$

It is known [14] that if $H(z) = \sum_{n=0}^{\infty} h_n z^n$ is analytic in E and $|H(z)| \le M$, then

$$\sum_{n=0}^{\infty} |h_n|^2 \le M^2. \tag{3.22}$$

Applying (3.22) to (3.21), we get

$$(1-\xi)^2 + \sum_{n=1}^{\infty}\left\{\frac{(c-1+\lambda n)(a)_n}{(c-1)_{n+1}}\right\}^2 |a_{p+n}|^2 < \eta^2;$$

that is,

$$\sum_{n=1}^{\infty}\left\{\frac{(c-1+\lambda n)(a)_n}{(c-1)_{n+1}}\right\}^2 |a_{p+n}|^2 < \frac{(A-B)^2}{1-B^2}. \tag{3.23}$$

Similarly,

$$\sum_{n=1}^{\infty}\left\{\frac{(c-1+\lambda n)(a)_n}{(c-1)_{n+1}}\right\}^2 |b_{p+n}|^2 < \frac{(A-B)^2}{1-B^2}. \tag{3.24}$$

Now, for $|z| = r < 1$, by applying Cauchy-Schwarz inequality we find that

$$\begin{aligned}
&\left|(1-\lambda)\frac{\mathcal{L}_p(a,c)h(z)}{z^p} + \lambda\frac{\mathcal{L}_p(a,c-1)h(z)}{z^p} - \xi\right|^2 \\
&= \left|(1-\xi) + \sum_{n=1}^{\infty}\left\{\frac{(c-1+\lambda n)(a)_n}{(c-1)_{n+1}}\right\}^2 a_{p+n}b_{p+n}z^n\right|^2 \\
&\leq (1-\xi)^2 + 2(1-\xi)\sum_{n=1}^{\infty}\left\{\frac{(c-1+\lambda n)(a)_n}{(c-1)_{n+1}}\right\}^2 |a_{p+n}||b_{p+n}|r^n \\
&\quad + \left|\sum_{n=1}^{\infty}\left\{\frac{(c-1+\lambda n)(a)_n}{(c-1)_{n+1}}\right\}^2 a_{p+n}b_{p+n}z^n\right|^2 \\
&\leq (1-\xi)^2 + 2(1-\xi)\left[\sum_{n=1}^{\infty}\left\{\frac{(c-1+\lambda n)(a)_n}{(c-1)_{n+1}}\right\}^2 |a_{p+n}|^2 r^n\right]^{1/2} \\
&\quad + \left[\sum_{n=1}^{\infty}\left\{\frac{(c-1+\lambda n)(a)_n}{(c-1)_{n+1}}\right\}^2 |b_{p+n}|^2 r^n\right]^{1/2} \\
&\quad + \left[\sum_{n=1}^{\infty}\left\{\frac{(c-1+\lambda n)(a)_n}{(c-1)_{n+1}}\right\}^2 |a_{p+n}|^2 r^n\right] + \left[\sum_{n=1}^{\infty}\left\{\frac{(c-1+\lambda n)(a)_n}{(c-1)_{n+1}}\right\}^2 |b_{p+n}|^2 r^n\right] \\
&\leq (1-\xi)^2 + 2(1-\xi)\left[\sum_{n=1}^{\infty}\left\{\frac{(c-1+\lambda n)(a)_n}{(c-1)_{n+1}}\right\}^2 |a_{p+n}|^2\right]^{1/2} \\
&\quad \times \left[\sum_{n=1}^{\infty}\left\{\frac{(c-1+\lambda n)(a)_n}{(c-1)_{n+1}}\right\}^2 |b_{p+n}|^2\right]^{1/2} \\
&\quad + \left[\sum_{n=1}^{\infty}\left\{\frac{(c-1+\lambda n)(a)_n}{(c-1)_{n+1}}\right\}^2 |a_{p+n}|^2\right] + \left[\sum_{n=1}^{\infty}\left\{\frac{(c-1+\lambda n)(a)_n}{(c-1)_{n+1}}\right\}^2 |b_{p+n}|^2\right] \\
&\leq (1-\xi)^2 + 2(1-\xi)\frac{(A-B)^2}{1-B^2} + \frac{(A-B)^4}{(1-B^2)^2} \\
&= \left\{\frac{B(A-B)}{1-B^2}\right\}^2 + 2\frac{B(A-B)^3}{(1-B^2)^2} + \frac{(A-B)^4}{(1-B^2)^2} = \frac{A^2(A-B)^2}{(1-B^2)^2} < \eta^2,
\end{aligned}$$

by using (3.23) and (3.24). Thus, again with the aid of (3.21), $h \in \mathcal{V}_p^{\lambda}(a,c,A,B)$.

THEOREM 8. *Let $f \in \mathcal{V}_p^{\lambda}(a,c,A,B)$ $(\lambda > 0)$ and $S_n(z) = z^p + \sum_{k=1}^{n-1} a_{p+k}z^{p+k}$ $(n \geq 2)$. Then for $z \in E$, we have*

$$\operatorname{Re}\left[\frac{\int_0^z t^{-p}\,(\mathcal{L}_p(a,c)S_n(t))\,dt}{z}\right] > \kappa(\lambda,c,A,B),$$

where $\kappa(\lambda,c,A,B)$ is defined as in Theorem 1.

PROOF. Singh and Singh [9] proved that

$$\operatorname{Re}\left\{1 + \sum_{k=1}^{n-1}\frac{z^k}{k+1}\right\} > \frac{1}{2} \qquad (z \in E). \tag{3.25}$$

Writing

$$\frac{\int_0^z t^{-p}\,\mathcal{L}_p(a,c)S_n(z)\,dt}{z} = \frac{\mathcal{L}_p(a,c)f(z)}{z^p} * \left\{1 + \sum_{k=1}^{n-1} \frac{z^k}{k+1}\right\}$$

and making use of (3.25), Theorem 1, and Lemma 3, the assertion of the theorem follows at once.

Taking $\lambda = 1$, $a = b = p+1$, $A = 1 - 2\alpha/p$, and $B = -1$ in Theorem 8, we get the following corollary.

COROLLARY 10. *If $f \in \mathcal{A}_p$ satisfies $\operatorname{Re}\{f'(z)/z^{p-1}\} > \alpha$ $(0 \le \alpha < p)$ in E, then*

$$\operatorname{Re}\left[\frac{\int_0^z t^{-p}\,S_n(t)\,dt}{z}\right] > \frac{\alpha}{p} + \left(1 - \frac{\alpha}{p}\right)\left\{{}_2F_1\left(1,1;p+1;\frac{1}{2}\right) - 1\right\} \qquad (z \in E).$$

REFERENCES

1. Z. Janowski, Some extremal problems for certain families of analytic functions, *Ann. Polon. Math.* **28**, 297–326, (1973).
2. H. Saitoh and M. Nunokawa, On certain subclasses of analytic functions involving a linear operator, *Sūrikaisekikenkyūsho Kōkyūroku*, No. 963, 97–109, (1996).
3. St. Ruscheweyh, New criteria for univalent functions, *Proc. Amer. Math. Soc.* **49**, 109–115, (1975).
4. S. Fukui, J.A. Kim and H.M. Srivastava, On certain subclass of univalent functions by some integral operators, *Math. Japonica* **50**, 359–370, (1999).
5. N.S. Sohi, A class of p-valent analytic functions, *Indian J. Pure Appl. Math.* **10** (7), 826–834, (1979).
6. M.L. Mogra, On a class of univalent functions whose derivatives have a positive real part, *Riv. Mat. Univ. Parma.* **7**, 163–172, (1981).
7. S. Owa, On the distortion theorems, I, *Kyungpook Math. J.* **18**, 53–59, (1978).
8. S.S. Miller and P.T. Mocanu, Differential subordinations and univalent functions, *Michigan Math. J.* **28**, 157–171, (1981).
9. R. Singh and S. Singh, Convolution properties of a class of starlike functions, *Proc. Amer. Math. Soc.* **108**, 145–152, (1989).
10. M. Abramowitz and S.A. Stegun (Editors), *Handbook of Mathematical Formulas, Graphs, and Mathematical Tables*, pp. 429–446, Dover, New York, (1971).
11. M. Obradovič, On certain inequalities for some regular functions in $|z| < 1$, *Int. J. Math. & Math. Sci.* **8**, 671–681, (1985).
12. V. Anh, K-fold symmetric starlike univalent functions, *Bull. Austral. Math. Soc.* **32**, 419–436, (1985).
13. F.R. Keogh and E.P. Merkes, A coefficient inequality for certain classes of analytic functions, *Proc. Amer. Math. Soc.* **20**, 8–12, (1969).
14. Z. Nehari, *Conformal Mapping*, McGraw Hill, New York, (1952).